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Introduction to Hardy Spaces

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Basic material from the theory of Hardy spaces is presented. The principle of positive energy representations is used as motivation to introduce these spaces.

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1. THE PRINCIPLE OF POSITIVE ENERGY REPRESENTATIONS

We consider only the two simplest cases.

Case 1. The unit sphere S^1 as the rotation group $\mathbb{T}$. In the chiral conformal field theory the rotations are symmetries. After the exceptionality of the point "infinity" of the light line is removed then it is topologically a sphere. The Irrep's of the rotation group are the characters

$$\zeta \rightarrow \zeta^n, \quad n \in \mathbb{Z},$$

and a strongly continuous unitary representation $U(\mathbb{T})$ on a Hilbert space $\mathcal{H}$ can be decomposed as

$$U(\zeta) = \sum_{n \in Z} \zeta^n E_n,$$

where the E_n are the isotypical projections for the label n. These labels are usually interpreted as "energy levels."

Case 2. The real line $\mathbb{R}$ considered as (time) translation group. The Irrep's are the characters

$$a \rightarrow e^{-iap}, \quad p \in \mathbb{R},$$

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labeled by points of $\mathbb{R}$, usually interpreted as energy values. A strongly continuous representation on $\mathcal{H}$ can be decomposed as

$$U(a) = \int_{\mathbb{R}} e^{-iap} E(dp),$$

where E(·) is a spectral measure on $\mathcal{H}$.

The so-called positive energy representations are of special interest, physically as well as mathematically.

In Case 1, a representation is called *positive* if

$$E_n = 0 \quad \text{for} \quad n < 0, \quad \text{i.e.,} \quad U(\zeta) = \sum_{n=0}^{\infty} \zeta^n E_n,$$

i.e., there is a lowest energy, for example $n = 0$ if $E_0 > 0$.

In Case 2, the definition of *positivity* reads

$$E([0, \infty)) = \mathbf{1}, \quad \text{i.e.,} \quad U(a) = \int_0^{\infty} e^{-iap} E(dp).$$

This is the usual positivity condition for the energy in mathematical models.

Now the simplest representations are the so-called regular ones which are not positive.

Case 1. Choose $\mathcal{H} := L^2(S^1)$. The canonical ONB is denoted by e_n, $e_n(z) := z^n$, $n \in \mathbb{Z}$. Then the functions of $\mathcal{H}$ are given by Fourier series

$$f = \sum_{n \in \mathbf{Z}} a_n e_n, \qquad \sum_{n \in \mathbf{Z}} |a_n|^2 < \infty.$$

The regular representation is defined by

$$(U(\zeta) f)(z) := f(\zeta z).$$

Then

$$U(\zeta) = \sum_{n \in \mathbf{Z}} \zeta^n E_n \quad \text{with } E_n := (e_n, \cdot) e_n,$$

the one-dimensional projection onto the subspace $\mathbb{C} e_n$.

Case 2. Choose $\mathcal{H} := L^2(\mathbb{R})$. The Fourier transformation on $\mathcal{H}$ is denoted by F,

$$(Ff)(p) = \hat{f}(p) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-ipx} f(x)\, dx.$$

Then

$$(F^{-1} g)(x) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-ixp} g(p)\, dp.$$

The regular representation is defined by

$$(U(a)f)(x) := f(x - a).$$

Then one obtains $\hat{U}(a) := FU(a)F^{-1}$ is a multiplication operator given by

$$(\hat{U}(a)\hat{f})(p) = e^{-iap}\hat{f}(p).$$

i.e., the spectral measure of $\hat{U}(\mathbb{R})$ is given by the spectral measure of the characteristic functions,

$$\hat{E}((\Delta)f)(p) = \chi\Delta(p)f(p), \qquad \Delta \subset \mathbb{R}.$$

Then the spectral measure $E(\cdot)$ is given by

$$E(\Delta) = F^{-1}\chi\Delta F.$$

It is easy to separate from the regular representation subrepresentations which are positive.

Case 1. Choose the subspace $H^2(S^1) \subset L^2(S^1)$, called *Hardy space* defined by

$$H^2(S^1) := \left\{ f \in L^2(S^1) : f = \sum_{n=0}^{\infty} a_n e_n \right\}.$$

Then $H^2(S^1)$ is invariant w.r.t. the regular representation and $U(\mathbb{T}) \upharpoonright H^2(S^1)$ is positive.

Case 2. Choose the subspace $H^2_+(\mathbb{R}) \subset L^2(\mathbb{R})$, called *Hardy space*, defined by

$$H^2_+(\mathbb{R}) := F^{-1}\chi_{[0,\infty)}L^2(\mathbb{R}) = F^{-1}L^2(\mathbb{R}_+), \qquad \mathbb{R}_+ := [0, \infty),$$

i.e., $f \in H^2_+(\mathbb{R})$ if $\hat{f}(p) \equiv 0$ (modulo Lebesgue measure) for $p < 0$. Then one has again: $H^2_+(\mathbb{R})$ is invariant w.r.t. the regular representation and $U(\mathbb{R}) \upharpoonright H^2_+(\mathbb{R})$ is positive.

In the following several interesting properties of the Hardy spaces are presented.

2. THE UNIT SPHERE S^1

Recal first that Fourier exansions of functions from $L^2(S^1)$ are "formally similar" to Laurent expansions in analytic function theory. Therefor the functions in $H^2(S^1)$ seem to be the "analytic elements" of L^2. More precisely, the first observation is the following: Let

$$H^2(S^1) \ni f = \sum_{n=0}^{\infty} \alpha_n e_n, \sum_{n=0}^{\infty} |\alpha_n|^2 < \infty.$$

Then the power series

$$\sum_{n=0}^{\infty} \alpha_n z^n,$$

has a radius of convergence $r \geq 1$ because

$$|\sum_{n=k+1}^{m} |\alpha_n| \cdot |z|^n| \leq \left(\sum_{n=k+1}^{m} |\alpha_n|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{n=k+1}^{m} |z|^{2n}\right)^{\frac{1}{2}} \leq \epsilon \cdot \left(\frac{1}{1-|z|^2}\right)^{\frac{1}{2}},$$

for $|z| < 1$ and sufficiently large k and defines therefore a holomorphic function

$$f(z) := \sum_{n=0}^{\infty} \alpha_n z^n, \qquad |z| < 1$$

in the open unit disc D of the complex plane (in some sense $f(\cdot)$ is the extension of f into the interior). It turns out that one has a bijection

$$H^2(S^1) \ni f \leftrightarrow f(\cdot) \in H^2(D),$$

between $H^2(S^1)$ and the set $H^2(D)$ of all holomorphic functions in D such that the sequence of Taylor coefficients is square-summable. It should be mentioned that

$$H^D(D) \subset A^2(D), \tag{1}$$

where $A^2(D)$ denotes the Hilbert space of all complex-valued functions that are holomorphic throughout D and square-integrable w.r.t. μ, the planar Lebesgue measure measure in D. The scalar product is given by

$$(f, g) := \int_D \overline{f(z)} g(z) \mu(dz)$$

and a canonical ONB is given by

$$e_n(z) := \left(\frac{n+1}{\pi}\right)^{1/2} z^n, \quad n = 0, 1, 2, \ldots,$$

so that $A^2(D)$ is isomorphic to the Hilbert space of all sequences $(\alpha_0, \alpha_1, \alpha_2, \ldots)$ with

$$\sum_{n=0}^{\infty} \frac{|\alpha_n|^2}{n+1} < \infty,$$

such that (1) follows.

It is easy to find a criterion for the square summability of the sequence of the Taylor coefficients. Choose a function ϕ, holomorphic in D, $\phi(z) := \sum_{n=0}^{\infty} \alpha_n z^n$.

Define

$$\phi_r(z) := \phi(rz), \qquad 0 < r < 1, \quad |z| = 1.$$

Then it is easy to see that first

$$\phi_r \in H^2(S^1), \quad 0 < r < 1,$$

holds and second

$$\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty \quad \text{iff} \quad \sup_{0<r<1} \|\phi\| < \infty.$$

The bijection of $H^2(S^1)$ and $H^2(D)$ given by the "extension of f into the interior" raises the question to the inverse problem: How can one determine f from $f(\cdot)$? The solution is simple:

Given $f(\cdot)$ define

$$f_r(z) := f(rz), \qquad 0 < r < 1, \quad |z| = 1.$$

Then obviously

$$f_r \in H^2(S^1), \qquad 0 < r < 1,$$

and the assertion is that

$$f = \text{s-lim}_{r\to 1} f_r,$$

in the sense of convergence in the norm of the Hilbert space $H^2(S^1)$. The corresponding argument is simple: Let $f = \sum_{n=0}^{\infty} \alpha_n e_n$. Then $f_r = \sum_{n=0}^{\infty} \alpha_n r^n e_n$ and

$$\| f - f_r \|^2 = \sum_{n=0}^{\infty} |\alpha_n|^2 (1 - r^n)^2 \leq \sum_{n=0}^{k} |\alpha_n|^2 (1 - r^n)^2 + \sum_{n=k+1}^{\infty} |\alpha_n|^2,$$

which implies the assertion. Moreover,

$$\| f \| = \lim_{r\to 1} \| f_r \| = \sup_{0<r<1} \| f_r \|$$

follows.

3. THE REAL LINE $\mathbb{R}$

Let $f \in H^2_+(\mathbb{R}) = F^{-1} L^2(\mathbb{R}_+)$, i.e.,

$$f(x) = (2\pi)^{-1/2} \int_0^{\infty} e^{ixp} \hat{f}(p)\, dp, \qquad \hat{f}(p) = 0, \quad p < 0.$$

Obviously, f can be analytically continued into the upper half plane. Put $z = x + iy$ and define

$$f(z) := (2\pi)^{-1/2} \int_0^\infty e^{i(x+iy)p} \hat{f}(p)\, dp = (2\pi)^{-1/2} \int_0^\infty e^{ixp} \cdot e^{-yp} \hat{f}(p)\, dp.$$

Then

$$f \text{ is holomorphic for } \mathfrak{I}z > 0 \tag{2}$$

and

$$f(\cdot + iy) \in L^2(\mathbb{R}), \tag{3}$$

because its Fourier transform $p \to e^{-yp} \hat{f}(p)$ is an L^2-function. Parseval's equation gives

$$\infty > \int_{\mathbb{R}} |f(x+iy)|^2\, dx = \int_0^\infty e^{-2yp} |\hat{f}(p)|^2\, dp, \quad y > 0.$$

Hence

$$\sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2\, dx < \infty \tag{4}$$

follows. Moreover, f appears as the boundary value for the function $f(\cdot + iy$ for $y \to +0$ (in the sense of norm convergence in $L^2(\mathbb{R})$).

Surprisingly, the properties (2), (3), (4) characterize $H^2_+(\mathbb{R})$ completely. This is the content of the famous theorem of Paley/Wiener (see, Raymond and Paley, 1934). A proof can also be found in the textbook of Yosida (1971).

Theorem *(Paley/Wiener). Let $f(\cdot)$ be a holomorphic function in the upper half plane. Further let $f(\cdot + iy) \in L^2(R)$ for each fixed $y > 0$, $z = x + iy$, and*

$$\sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2\, dx < \infty.$$

Then

$$f \in H^2_+(\mathbb{R})$$

follows.

The strategy of the proof is to show first that $f(\cdot + iy)$ is a Cauchy sequence for $y \to +0$ w.r.t. the L^2-norm. Thus there is a limit function

$$L^2(\mathbb{R}) \ni f(\cdot + i0) = \text{s-lim}_{y\to+0} f(\cdot + iy),$$

i.e.,

$$\lim_{y\to+0} \int_{\mathbb{R}} |f(x+iy) - f(x+i0)|^2\, dx = 0.$$

Second, it turns out that the Fourier transform

$$\hat{f}_{+0} := Ff(\cdot + i0) \in L^2(\mathbb{R})$$

vanishes on the negative half line, so that

$$f(z) = (2\pi)^{-1/2} \int_0^\infty e^{izp} \hat{f}_{+0}(p)\, dp.$$

Interestingly enough, an element $f \in H^2_+(\mathbb{R})$ is uniquely determined by its projection onto an arbitrary open interval $(a, b) \subset \mathbb{R}$, where $a = -\infty$, $b < \infty$ and $-\infty < a$, $b = \infty$ are allowed. For example, the case $(0, \infty)$, the positive half line, is often used in applications.

Theorem. *Let $f \in H^2_+(\mathbb{R})$ and $f(x + i0) = 0$ a.e. for $x > 0$. Then $f(z) = 0$ for all z from the upper half plane.*

The proof uses Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad \mathfrak{L}z > 0,$$

where $\mathcal{C}$ is a positively oriented rectangular path and z is contained in its interior. Then, by standard limit processes (e.g., see Yosida, 1971) one gets

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\lambda + i0)}{\lambda - z}\, d\lambda, \quad \mathfrak{L}z > 0.$$

On the other hand, one has

$$\int_{-\infty}^{\infty} \frac{f(\lambda + i0)}{\lambda - z}\, d\lambda = 0, \qquad \mathfrak{L}z < 0.$$

Now, if $(x + i0) = 0$ a.e. for $x > 0$, the integral

$$\frac{1}{2\pi i} \int_{-\infty}^{0} \frac{f(\lambda + i0)}{\lambda - z}\, d\lambda$$

defines a function of z which is holomorphic in the region $\mathbb{C}\backslash(-\infty, 0]$. This implies immediately $f(z) = 0$ for $\mathfrak{L}z > 0$.

4. HILBERT TRANSFORM

A beautiful application of this theorem concerns the so-called Hilbert transform. Start with a so-called Cauchy integral

$$f(z) := \frac{1}{i\pi} \int_{\mathbb{R}} \frac{1}{\lambda - z} \phi(\lambda)\, d\lambda, \quad \phi \in L^2(\mathbb{R}), \quad \mathfrak{L}z > 0.$$

Define

$$f_\epsilon(x) := f(x+i\epsilon) = -\frac{1}{i\pi}\int_{\mathbb{R}} \frac{1}{x+i\epsilon-\lambda}\phi(\lambda)\,d\lambda.$$

Then one has $f_\epsilon \in L^2(\mathbb{R})$ for all $\epsilon > 0$. Note that

$$f_\epsilon = -\frac{1}{i\pi}\left(\frac{1}{\cdot + i\epsilon} * \phi\right).$$

Therefore, according to the convolution theorem,

$$\begin{aligned}\hat{f}_\epsilon(p) = F(F_\epsilon)(p) &= -\frac{1}{i\pi}(2\pi)^{1/2}F\left(\frac{1}{\cdot+i\epsilon}\right)(p)\cdot F(\phi)(p)\\ &= -\frac{1}{i\pi}(2\pi)^{1/2}(-(2\pi)^{1/2}\, i\, e^{-\epsilon p}\chi_{[0,\infty)}(p))\hat{\phi}(p)\\ &= 2\chi_{[0,\infty)}(p)\, e^{-\epsilon}\hat{\phi}(p),\end{aligned}$$

hence

$$\|\hat{f}_\epsilon\|_{L^2} \leq 2\|\phi\|_{L^2}, \quad \epsilon > 0,$$

follows, i.e., $f_\epsilon H^2_+(\mathbb{R})$. Therefore, according to the Paley/Wiener theorem, the limit function exists

$$f_{+0} := \text{s-lim}_{\epsilon\to+0} f_\epsilon.$$

Now f_{+0} can be calculated explicity. One has $f(x+i\epsilon) = g_\epsilon(x) + h_\epsilon(x)$, where

$$g_\epsilon(x) = -\frac{1}{i\pi}\int_{\mathbb{R}} \frac{x-\lambda}{(x-\lambda)^2+\epsilon^2}\phi(\lambda)\,d\lambda,$$

$$h_\epsilon(x) = \frac{\epsilon}{\pi}\int_{\mathbb{R}} \frac{1}{(x-\lambda)^2+\epsilon^2}\phi(\lambda)\,d\lambda$$

and finally

$$f_{+0}(x) = (H\phi)(x) + \phi(x),$$

where

$$(H\phi)(x) := \frac{1}{i\pi}\int_{\mathbb{R}} \frac{\phi(\lambda)}{\lambda - x}d\lambda \qquad \text{(Cauchy's main value)}$$

is the Hilbert transform. Hence

$$2\chi_{[0,\infty}(p)\hat{\phi}(p) = (Ff_{+0})(p) = F(H(\phi))(p) + \hat{\phi}(p),$$

and

$$F(H(\phi))(p) = (2\chi_{[0,\infty}(p) - 1)\hat{\phi}(p) = \text{sgn}\, p\cdot\hat{\phi}(p),$$

follow and this means

$$((FHF^{-1})\hat{\phi}(p) = \operatorname{sgn} p \cdot \hat{\phi}(p),$$

so that H is self-adjoint and idempotent, $H = H^*$, $H^2 = \mathbf{1}$.

5. RESOLVENT LIMITS

A further application of Hardy spaces is concerned with the problem of the existence of limits $\epsilon \to 0$ for resolvents

$$R(z) := (z\mathbf{1} - H)^{-1}, \quad \mathfrak{I}z \neq 0,$$

of a self-adjoint operator $H = \int_{\mathbb{R}} \lambda E(d\lambda)$ on a Hilbert space H, where $z = x + i\epsilon$. More precisely, one studies matrix elements $(u, R(z)v)$, $u, v \in \mathcal{H}$ or vector functions $AR(z) f$ for certain bounded operators on $\mathcal{H}$ and for special vectors $f \in \mathcal{H}$.

First note that it is straightforward to generalize the concept "Hardy space": replace $\mathbb{C}$ by a Hilbert space $\mathcal{H}$. Then the underlying Hilbert space is now $L^2(\mathbb{R}, \mathcal{H})$ and the corresponding Hardy space is denoted by $H^2_+(\mathbb{R}, \mathcal{H})$, correspondig to $L^2(\mathbb{R}_+, \mathcal{H})$.

Second, replace the upper half plane by the lower half plane and the positive real half line $\mathbb{R}_+ = [0, \infty)$ by the negative real half line $\mathbb{R}_- = (-\infty, 0]$. Then for all these generalized concepts the former arguments can be repeated and lead to the Hardy space $H^2_-(\mathbb{R}, \mathcal{H})$, corresponding to $L^2_+(\mathbb{R}_-, \mathcal{H})$. Note that H^2_+ and H^2_- are mutually orthogonal and one has the orthogonal decompositions

$$L^2(\mathbb{R}_-, \mathcal{H}) \oplus L^2(\mathbb{R}_+, \mathcal{H}) = L^2(\mathbb{R}, \mathcal{H}) = H^2_+(\mathbb{R}, \mathcal{H}) \oplus H^2_-(\mathbb{R}, \mathcal{H}). \tag{5}$$

To present the mentioned application start with the identity

$$i(\lambda + i\epsilon - H)^{-1} = \int_0^\infty e^{i((\lambda+i\epsilon)-H)x}\, dx, \quad \epsilon > 0. \tag{6}$$

Applying (6) on a vector f and multiplying from the left by a bounded operator A, then by Parseval's equation one obtains

$$\frac{1}{2\pi} \int_{\mathbb{R}} \|AR(\lambda + i\epsilon) f\|^2\, d\lambda = \int_0^\infty e^{-2\epsilon t} \|A\, e^{-itH} f\|^2\, dt \tag{7}$$

for all $f \in \mathcal{H}$, and, correspondingly,

$$\frac{1}{2\pi} \int_{\mathbb{R}} \|AR(\lambda - i\epsilon) f\|^2\, d\lambda = \int_{-\infty}^0 e^{2\epsilon t} \|A\, e^{-itH} f\|^2\, dt. \tag{8}$$

For the application of the Hardy spaces in this framework it is necessary to restrict the consideration to the so-called absolutely continuous subspace of H. Therefore, for convenience, we assume that H itself is already absolutely continuous. This means that for each $f \in \mathcal{H}$, $\| f \| = 1$, the measure $(f, E(\Delta f))$ is absolutely continuous w.r.t. the Lebesgue measure.

An important dense submanifold $\mathcal{M}_\infty \subset \mathcal{H}$ is given as follows: Put

$$| f |_\infty := \left(\text{ess sup}_\lambda \frac{(f, E(d\lambda) f)}{d\lambda} \right)^{1/2}.$$

Then $\mathcal{M}_\infty$ is defined to consist of all f with $| f |_\infty < \infty$.

The relations (7) and (8) are of interest in particular for Hilbert–Schmidt operators $A \in \mathcal{L}_2(\mathcal{H})$ and for vectors $f \in \mathcal{M}_\infty$. Namely, the following Lemma is true:

Lemma 3. *Let $A \in \mathcal{L}_2(\mathcal{H})$ and $f \in \mathcal{M}_\infty$. Then*

$$\int_{\mathbb{R}} \| A e^{-itH} f \|^2 \, dt \leq 2\pi \| A \|_2^2 \cdot | f |_\infty^2,$$

where $\| A \|_2$ denotes the Hilbert-Schmidt norm of A.

Obviously this implies

$$\sup_{\epsilon > 0} \int_{\mathbb{R}} \| AR(\lambda + i\epsilon) f \|^2 \, d\lambda \leq 2\pi \int_0^\infty \| A e^{-itH} f \|^2 \, dt,$$

and

$$\sup_{\epsilon > 0} \int_{\mathbb{R}} \| AR(\lambda - i\epsilon) f \|^2 \, d\lambda \leq 2\pi \int_{-\infty}^0 \| A e^{-itH} f \|^2 \, dt.$$

Therefore Paley/Wiener's theorem is applicable and one obtains the existence of the strong limits

$$\text{s-lim}_{\epsilon \to 0} AR(\lambda \pm i\epsilon) f =: AR(\lambda \pm i0) f,$$

and the statement on their Fourier transforms. Now recall that

$$\frac{1}{2i\pi}(R(\lambda - i\epsilon) - R(\lambda + i\epsilon)) = \int_{\mathbb{R}} \delta_\epsilon(\lambda - x) E(dx),$$

where

$$\bar{\delta}_\epsilon(x) := \frac{\epsilon}{\pi} \cdot \frac{1}{x^2 + \epsilon^2}.$$

A further observation says that

$$\frac{AE(d\lambda) f}{d\lambda}$$

exists almost everywhere, this function is a member of $L^2(\mathbb{R}, \mathcal{H})$ and

$$\frac{AE(d\lambda)f}{d\lambda} = \text{s-lim}_{\epsilon\to 0} \int_{\mathbb{R}} \delta_\epsilon(\lambda - x) AE(dx) f.$$

Therefore, finally one gets

$$\frac{AE(d\lambda)f}{d\lambda} = \frac{1}{2i\pi}(AR(\lambda - i0)f - AR(\lambda + i0)f),$$

which is the decomposition of the left-hand side w.r.t. the decomposition (5).

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